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Abstract
A new representation of the $2N$-fold integrals appearing in various two-matrix models that admit reductions to integrals over their eigenvalues is given in terms of vacuum state expectation values of operator products formed from two-component free fermions. This is used to derive the perturbation series for these integrals under deformations induced by exponential weight factors in the measure, expressed as double and quadruple Schur function expansions, generalizing results obtained earlier for certain two-matrix models. Links with the coupled two-component KP hierarchy and the two-component Toda lattice hierarchy are also derived.

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1. Introduction

Let $d\mu(x, y)$ be a measure (in general, complex), supported either on a finite set of products of curves in the complex x and y planes or, alternatively, on a domain in the complex z plane, with the identifications $(x = z, y = \bar{z})$. Consider the following $2N$-fold integral

$$Z_N = \int d\mu(x_1, y_1) \ldots \int d\mu(x_N, y_N) \Delta_N(x) \Delta_N(y), \tag{1.1}$$

evaluated over some finite linear combination of support domains, where

$$\Delta_N(x) = \prod_{i<j}^{N}(x_i - x_j), \qquad \Delta_N(y) = \prod_{i<j}^{N}(y_i - y_j) \tag{1.2}$$

are Vandermonde determinants.

Such integrals arise, in particular, as partition functions and correlation functions for two-matrix models in those cases where one can reduce the integration over the matrix ensemble, e.g. via the Itzykson–Zuber, Harish–Chandra identity [14], to integrals over the eigenvalues $\{x_i, y_i\}_{i=1,\dots,N}$ of the two matrices. Depending on the specific choice of measure and support, these include models of normal matrices [5] with spectrum supported on some open region of the complex plane, coupled pairs of random hermitian matrices [14, 20] or, more generally, normal matrices with spectrum supported on curve segments in the complex plane [3, 4], and certain models of random unitary matrices [28, 29] (see appendix A for several examples).

From the viewpoint of perturbation theory, and also to apply methods from the theory of integrable systems to study such integrals, it is of interest to consider deformations of the measure of the general form

$$\begin{aligned}
d\mu(x, y|\mathbf{t}, n, m, \bar{\mathbf{t}}) := x^n y^m \exp(V(x, \mathbf{t}^{(1)}) + V(y, \mathbf{t}^{(2)}) \\
+ V(x^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y^{-1}, \bar{\mathbf{t}}^{(2)})) \, d\mu(x, y),
\end{aligned} \tag{1.3}$$

where

$$V(x, \mathbf{t}^{(\alpha)}) = \sum_{k=1}^{\infty} t_k^{(\alpha)} x^k, \qquad V(x^{-1}, \bar{\mathbf{t}}^{(\alpha)}) = \sum_{k=1}^{\infty} \bar{t}_k^{(\alpha)} x^{-k}, \quad \alpha = 1, 2. \tag{1.4}$$

The four infinite sequences of complex numbers $\mathbf{t}^{(1)} = (t_1^{(1)}, t_2^{(1)}, \dots)$, $\mathbf{t}^{(2)} = (t_1^{(2)}, t_2^{(2)}, \dots)$, $\bar{\mathbf{t}}^{(1)} = (\bar{t}_1^{(1)}, \bar{t}_2^{(1)}, \dots)$, $\bar{\mathbf{t}}^{(2)} = (\bar{t}_1^{(2)}, \bar{t}_2^{(2)}, \dots)$ and the integers n, m are viewed as independent deformation parameters. (Thus the quantities $(\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$ are not, in general, complex conjugates of $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$.) For brevity, we also use the notation $\mathbf{t} := (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$, $\bar{\mathbf{t}} := (\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$. The correspondingly deformed Z_N will be denoted $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$.

In this work we represent the integral (1.1) and its deformations as vacuum state expectation values (VEV) using two-component fermions, and use this to derive perturbation expansions for $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ as series in products of two or four Schur functions, with the four sets of continuous deformation parameters $\{\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}\}$ as their arguments. Such character expansions have been studied for various special cases of one- and two-matrix models by a number of authors [2, 9, 10, 17, 18, 23, 28, 29] using a variety methods. The fermionic approach presented here is based on the original constructions of τ-functions for integrable hierarchies ([6, 15, 16]) as fermionic VEVs.

In section 2, after presenting the integral (1.1) in the form of a fermionic vacuum expectation value, we show that the corresponding integral $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ for the deformed measure (1.3) is a special case of a τ-function for the coupled two-component KP hierarchy or, equivalently, a τ-function for the two-component Toda lattice (TL) hierarchy [6, 24–26] (see (2.15)).

We also use this fermionic representation to derive the perturbation expansion for $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ as weighted power series in the deformation parameters, written in the following alternative forms as sums over products of Schur functions

$$Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \sum_{\lambda, \mu, \nu, \eta} I_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(\bar{\mathbf{t}}^{(2)}) \tag{1.5}$$

$$= N! \sum_{\lambda, \mu} g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) \tag{1.6}$$

$$= N! \sum_{\lambda,\mu} g_{\lambda\mu}^{--}(N, n, m, \mathbf{t}^{(1)}, \mathbf{t}^{(2)}) s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}) \tag{1.7}$$

$$= N! \sum_{\lambda,\mu} g_{\lambda\mu}^{+-}(N, n, m, \bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}) \tag{1.8}$$

$$= N! \sum_{\lambda,\mu} g_{\lambda\mu}^{-+}(N, n, m, \mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)}) s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{1.9}$$

where $s_\lambda(\mathbf{t})$ is the Schur function corresponding to a partition $\lambda := (\lambda_1, \ldots, \lambda_N)$ of length $N = \ell(\lambda)$ (see [19], or appendix B) and the sums are over all quadruples $(\lambda, \mu, \nu, \eta)$ or pairs (λ, μ) of such partitions. The coefficients $I_{\lambda\mu\nu\eta}(N, n, m)$, $g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$, $g_{\lambda\mu}^{--}(N, n, m, \mathbf{t}^{(1)}, \mathbf{t}^{(2)})$, $g_{\lambda\mu}^{+-}(N, n, m, \bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)})$ and $g_{\lambda\mu}^{-+}(N, n, m, \mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)})$ are expressed as determinants of $N \times N$ submatrices of the matrix of bimoments

$$B_{ik} = \int x^i y^k \, \mathrm{d}\mu(x, y), \quad i, k \in \mathbf{Z} \tag{1.10}$$

and its deformations (see equations (2.31)–(2.35) and (3.19)).

For normal matrix models, formula (1.6) was presented in [23]. The case of diagonal coefficients $g_{\lambda\mu} = \delta_{\lambda\mu} r_\lambda$ was earlier considered in [9, 10] in the context of models of pairs of hermitian matrices with Itzykson–Zuber coupling [14] and of normal matrices with axially symmetric interactions [5]. In this work, we derive formulae (1.5)–(1.9) by two different methods. First, in section 2 via the two-component fermionic calculus and then, in section 3, by a direct calculation, as in [9, 10, 23], based on a standard determinant integral identity (the Andréief formula (3.1)), combined with the Cauchy–Littlewood identity (3.3). We also present some of the earlier results in a more complete way; in particular, the expansions (1.6)–(1.9) are given for the case of coupled pairs of hermitian matrices. The fermionic representation to be used here however differs from the ones appearing previously in the context of matrix models in [9, 10, 27], where the links with the one component KP (and TL) hierarchy were developed. It is also different from the fermionic approach of [5], where two-dimensional fermions were used in the study of normal matrix models.

Remark 1.1. One can obtain the N-fold integrals arising in one-matrix models as specializations of the above, either by choosing the undeformed measure to contain a factor proportional to the Dirac delta function $\delta(x - y)$, in which case the matrix of bimoments becomes a Hankel matrix, or to $\delta\left(x - \frac{1}{y}\right)$, in which case it becomes a Toeplitz matrix. We shall not consider these specializations here.

1.1. Free fermions

Let $\mathcal{A}$ be the complex Clifford algebra over $\mathbb{C}$ generated by *charged free fermions* $\{f_i, \bar{f}_i\}_{i \in \mathbf{Z}}$, satisfying the anticommutation relations

$$[f_i, f_j]_+ = [\bar{f}_i, \bar{f}_j]_+ = 0, \qquad [f_i, \bar{f}_j]_+ = \delta_{ij}. \tag{1.11}$$

Any element of the linear part

$$W := (\oplus_{m \in \mathbf{Z}} \mathbb{C} f_m) \oplus (\oplus_{m \in \mathbf{Z}} \mathbb{C} \bar{f}_m) \tag{1.12}$$

will be referred to as a *free fermion*. We also introduce the fermionic free fields

$$f(x) := \sum_{k \in \mathbf{Z}} f_k x^k, \qquad \bar{f}(y) := \sum_{k \in \mathbf{Z}} \bar{f}_k y^{-k-1}, \tag{1.13}$$

which may be viewed as generating functions for the $f_j, \bar{f}_j$'s.

This Clifford algebra has a standard Fock space representation defined as follows. Define the complementary, totally null (with respect to the underlying quadratic form) and mutually dual subspaces

$$W_{an} := (\oplus_{m<0} \mathbb{C} f_m) \oplus (\oplus_{m \geqslant 0} \mathbb{C} \bar{f}_m), \qquad W_{cr} := (\oplus_{m \geqslant 0} \mathbb{C} f_m) \oplus (\oplus_{m<0} \mathbb{C} \bar{f}_m), \qquad (1.14)$$

and consider the left and right $\mathcal{A}$-modules

$$F := \mathcal{A}/\mathcal{A} W_{an}, \qquad \bar{F} := W_{cr} \mathcal{A} \backslash \mathcal{A}. \qquad (1.15)$$

These are cyclic $\mathcal{A}$-modules generated by the vectors

$$|0\rangle = 1 \bmod \mathcal{A} W_{an}, \qquad \langle 0| = 1 \bmod W_{cr} \mathcal{A}, \qquad (1.16)$$

respectively, with the properties

$$\begin{aligned} f_m|0\rangle = 0 \quad (m < 0), \qquad \bar{f}_m|0\rangle = 0 \quad (m \geqslant 0), \\ \langle 0|f_m = 0 \quad (m \geqslant 0), \qquad \langle 0|\bar{f}_m = 0 \quad (m < 0). \end{aligned} \qquad (1.17)$$

The *Fock spaces* F and $\bar{F}$ are mutually dual, with the hermitian pairing defined via the linear form $\langle 0 || 0\rangle$ on $\mathcal{A}$, called the *vacuum expectation value*. This is determined by

$$\langle 0|1|0\rangle = 1; \qquad \langle 0|f_m \bar{f}_m|0\rangle = 1, \quad m < 0; \qquad \langle 0|\bar{f}_m f_m|0\rangle = 1, \quad m \geqslant 0, \qquad (1.18)$$

$$\langle 0|f_n|0\rangle = \langle 0|\bar{f}_n|0\rangle = \langle 0|f_m f_n|0\rangle = \langle 0|\bar{f}_m \bar{f}_n|0\rangle = 0; \quad \langle 0|f_m \bar{f}_n|0\rangle = 0, \quad m \neq n, \qquad (1.19)$$

together with the Wick theorem which implies, for any finite set of elements $\{w_k \in W\}$,

$$\langle 0|w_1 \cdots w_{2n+1}|0\rangle = 0,$$
$$\langle 0|w_1 \cdots w_{2n}|0\rangle = \sum_{\sigma \in S_{2n}} \text{sgn} \, \sigma \, \langle 0|w_{\sigma(1)} w_{\sigma(2)}|0\rangle \cdots \langle 0|w_{\sigma(2n-1)} w_{\sigma(2n)}|0\rangle. \qquad (1.20)$$

Here σ runs over permutations for which $\sigma(1) < \sigma(2), \ldots, \sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$.

Now let $\{w_i\}_{i=1,\ldots,N}$, be linear combinations of the f_j's only, $j \in \mathbf{Z}$, and $\{\bar{w}_i\}_{i=1,\ldots,N}$ linear combinations of the $\bar{f}_j$'s, $j \in \mathbf{Z}$. Then (1.20) implies

$$\langle 0|w_1 \cdots w_N \bar{w}_N \cdots \bar{w}_1|0\rangle = \det(\langle 0|w_i \bar{w}_j|0\rangle)|_{i,j=1,\ldots,N}. \qquad (1.21)$$

Following [6, 15], for all $N \in \mathbf{Z}$, we also introduce the states

$$\langle N| := \langle 0|C_N, \qquad (1.22)$$

where

$$C_N := \bar{f}_0 \cdots \bar{f}_{N-1}^{(\alpha)} \qquad \text{if} \quad N > 0 \qquad (1.23)$$

$$C_N := f_{-1} \cdots f_N \qquad \text{if} \quad N < 0 \qquad (1.24)$$

$$C_N := 1 \qquad \text{if} \quad N = 0 \qquad (1.25)$$

and

$$|N\rangle := \bar{C}_N|0\rangle, \qquad (1.26)$$

where

$$\bar{C}_N := f_{N-1} \cdots f_0 \qquad \text{if} \quad N > 0 \qquad (1.27)$$

$$\bar{C}_N := \bar{f}_N \cdots \bar{f}_{-1} \qquad \text{if} \quad N < 0 \qquad (1.28)$$

$$\bar{C}_N := 1 \qquad \text{if} \quad N = 0. \qquad (1.29)$$

The states (1.22) and (1.26) are referred to as the left and right charged vacuum vectors, respectively, with charge N.

In what follows we use the notational convention

$$\Delta_N(x) = \det\left(x_i^{N-k}\right)\big|_{i,k=1,\dots,N} \quad (N > 0), \qquad \Delta_0(x) = 1, \qquad \Delta_N(x) = 0 \quad (N < 0). \tag{1.30}$$

From the relations

$$\langle 0|\bar{f}_{N-k} f(x_i)|0\rangle = x_i^{N-k}, \qquad \langle 0|f_{-N+k-1}\bar{f}(y_i)|0\rangle = y_i^{N-k}, \quad k = 1, 2, \dots, \tag{1.31}$$

and (1.21), it follows that

$$\langle N|f(x_1)\cdots f(x_n)|0\rangle = \delta_{n,N}\Delta_N(x), \quad N \in \mathbf{Z}, \tag{1.32}$$

$$\langle -N|\bar{f}(y_1)\cdots\bar{f}(y_n)|0\rangle = \delta_{n,N}\Delta_N(y), \quad N \in \mathbf{Z}. \tag{1.33}$$

1.2. Two-component fermions

The two-component fermion formalism is obtained by relabelling the above as follows:

$$f_n^{(\alpha)} := f_{2n+\alpha-1}, \qquad \bar{f}_n^{(\alpha)} := \bar{f}_{2n+\alpha-1}, \tag{1.34}$$

$$f^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^k f_k^{(\alpha)}, \qquad \bar{f}^{(\alpha)}(z) := \sum_{k=-\infty}^{+\infty} z^{-k-1} \bar{f}_k^{(\alpha)}, \tag{1.35}$$

where $\alpha = 1, 2$. Then (1.11) is equivalent to

$$\left[f_n^{(\alpha)}, f_m^{(\beta)}\right]_+ = \left[\bar{f}_n^{(\alpha)}, \bar{f}_m^{(\beta)}\right]_+ = 0, \qquad \left[f_n^{(\alpha)}, \bar{f}_m^{(\beta)}\right]_+ = \delta_{\alpha,\beta}\delta_{nm}. \tag{1.36}$$

We denote the right and left vacuum vectors respectively as

$$|0, 0\rangle := |0\rangle, \qquad \langle 0, 0| := \langle 0|. \tag{1.37}$$

Relations (1.17) then become, for $\alpha = 1, 2$,

$$f_m^{(\alpha)}|0, 0\rangle = 0 \quad (m < 0), \qquad \bar{f}_m^{(\alpha)}|0, 0\rangle = 0 \quad (m \geqslant 0), \tag{1.38}$$

$$\langle 0, 0|f_m^{(\alpha)} = 0 \quad (m \geqslant 0), \qquad \langle 0, 0|\bar{f}_m^{(\alpha)} = 0 \quad (m < 0). \tag{1.39}$$

As in [6, 15], we also introduce the states

$$\langle n^{(1)}, n^{(2)}| := \langle 0, 0|C_{n^{(1)}} C_{n^{(2)}}, \tag{1.40}$$

where

$$C_{n^{(\alpha)}} := \bar{f}_0^{(\alpha)} \cdots \bar{f}_{n^{(\alpha)}-1}^{(\alpha)} \qquad \text{if} \quad n^{(\alpha)} > 0 \tag{1.41}$$

$$C_{n^{(\alpha)}} := f_{-1}^{(\alpha)} \cdots f_{n^{(\alpha)}}^{(\alpha)} \qquad \text{if} \quad n^{(\alpha)} < 0 \tag{1.42}$$

$$C_{n^{(\alpha)}} := 1 \qquad \text{if} \quad n^{(\alpha)} = 0. \tag{1.43}$$

and

$$|n^{(1)}, n^{(2)}\rangle := \bar{C}_{n^{(2)}} \bar{C}_{n^{(1)}}|0, 0\rangle \tag{1.44}$$

where

$$\bar{C}_{n^{(\alpha)}} := f_{n^{(\alpha)}-1}^{(\alpha)} \cdots f_0^{(\alpha)} \qquad \text{if} \quad n^{(\alpha)} > 0 \tag{1.45}$$

$$\bar{C}_{n^{(\alpha)}} := \bar{f}_{n^{(\alpha)}}^{(\alpha)} \cdots \bar{f}_{-1}^{(\alpha)} \qquad \text{if} \quad n^{(\alpha)} < 0 \tag{1.46}$$

$$\bar{C}_{n^{(\alpha)}} := 1 \qquad \text{if} \quad n^{(\alpha)} = 0. \tag{1.47}$$

The states (1.40) and (1.44) will be referred to, respectively, as left and right charged vacuum vectors with charges $(n^{(1)}, n^{(2)})$.

It is easily verified that

$$f_m^{(1)}|n, *\rangle = 0 \quad (m < n), \qquad \bar{f}_m^{(1)}|n, *\rangle = 0 \quad (m \geqslant n), \tag{1.48}$$

$$\langle n, *|f_m^{(1)} = 0 \quad (m \geqslant n), \qquad \langle n, *|\bar{f}_m^{(1)} = 0 \quad (m < n), \tag{1.49}$$

and similarly

$$f_m^{(2)}|*, n\rangle = 0 \quad (m < n), \qquad \bar{f}_m^{(2)}|*, n\rangle = 0 \quad (m \geqslant n), \tag{1.50}$$

$$\langle *, n|f_m^{(2)} = 0 \quad (m \geqslant n), \qquad \langle *, n|\bar{f}_m^{(2)} = 0 \quad (m < n). \tag{1.51}$$

Remark 1.2. In subsequent calculations we use Wick's theorem in the form (1.21). In the two-component setting we use formula (1.21) separately for each component. To calculate the vacuum expectation value of an operator O, first express it in the form

$$O = \sum_i O_i^{(1)} O_i^{(2)}. \tag{1.52}$$

Then

$$\langle 0, 0|O|0, 0\rangle = \sum_i \langle 0, 0|O_i^{(1)} O_i^{(2)}|0, 0\rangle = \sum_i \langle 0, 0|O_i^{(1)}|0, 0\rangle \langle 0, 0|O_i^{(2)}|0, 0\rangle \tag{1.53}$$

where Wick's theorem in the form (1.21) may be applied to each factor $\langle 0, 0|O_i^{(\alpha)}|0, 0\rangle$.

As a first application, note that Wick's theorem and (1.32)–(1.33) imply that

$$\langle N, -N| \prod_{i=1}^{k} f^{(1)}(x_i) \bar{f}^{(2)}(y_i)|0, 0\rangle = (-1)^{\frac{1}{2}N(N+1)} \delta_{k,N} \Delta_N(x) \Delta_N(y), \tag{1.54}$$

where $\Delta_N(x)$ is defined in (1.30). It then easily follows that

$$\langle N + n, -N - m| \prod_{i=1}^{k} f^{(1)}(x_i) \bar{f}^{(2)}(y_i)|n, -m\rangle$$

$$= (-1)^{\frac{1}{2}N(N+1)} \delta_{k,N} \Delta_N(x) \Delta_N(y) \prod_{i=1}^{N} x_i^n (-y_i)^m. \tag{1.55}$$

2. Fermionic representation for $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ and double Schur function expansions

2.1. Fermionic representation for Z_N

Let

$$A := \int f^{(1)}(x) \bar{f}^{(2)}(y) \, d\mu(x, y). \tag{2.1}$$

Then the following gives a fermionic VEV representation of Z_N

$$Z_N = (-1)^{\frac{1}{2}N(N+1)} \langle N, -N|A^N|0, 0\rangle. \tag{2.2}$$

To see this, just use (1.54) to write

$$\langle N, -N | A^N | 0, 0 \rangle = \langle N, -N | \prod_{i=1}^{N} \int f^{(1)}(x_i) \bar{f}^{(2)}(y_i) \, \mathrm{d}\mu(x_i, y_i) | 0, 0 \rangle \tag{2.3}$$

$$= (-1)^{\frac{1}{2} N(N+1)} \int \cdots \int \Delta_N(x) \Delta_N(y) \prod_{i=1}^{N} \mathrm{d}\mu(x_i, y_i), \tag{2.4}$$

where the last equality follows from (1.54). Now introduce

$$g := \mathrm{e}^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}. \tag{2.5}$$

Since

$$\langle N, -N | A^k | 0, 0 \rangle = 0, \qquad \text{if} \quad k \neq N, \tag{2.6}$$

we may equivalently express (2.2) as

$$Z_N = (-1)^{\frac{1}{2} N(N+1)} N! \langle N, -N | g | 0, 0 \rangle, \quad N \geqslant 0. \tag{2.7}$$

By (1.54) we also obtain

$$\langle N, -N | \mathrm{e}^A | 0, 0 \rangle = 0, \quad N < 0. \tag{2.8}$$

Next we show that for the deformed measure (1.3) the expression (2.7) (or, equivalently (2.2)) determines a τ-function in the sense of integrable hierarchies.

2.2. $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ as a τ-function [6]

We begin by recalling briefly the important notion of τ-functions as introduced by the Sato school (see [6, 15]). First, we define two infinite linear families of operators $H(\mathbf{t})$, $\bar{H}(\bar{\mathbf{t}})$ by

$$H(\mathbf{t}) := \sum_{k=1}^{\infty} H_k^{(1)} t_k^{(1)} - \sum_{k=1}^{\infty} H_k^{(2)} t_k^{(2)}, \qquad \bar{H}(\bar{\mathbf{t}}) := \sum_{k=1}^{\infty} H_{-k}^{(2)} \bar{t}_k^{(2)} - \sum_{k=1}^{\infty} H_{-k}^{(1)} \bar{t}_k^{(1)}, \tag{2.9}$$

where the 'commuting Hamiltonians' $H_k^{(\alpha)}$ are bilinear combinations of fermion components of the form

$$H_k^{(\alpha)} := \sum_{n=-\infty}^{+\infty} f_n^{(\alpha)} \bar{f}_{n+k}^{(\alpha)}, \quad k \neq 0, \quad \alpha = 1, 2. \tag{2.10}$$

Now, let

$$g := \exp \sum_{\alpha, \beta = 1, 2} \int \; : f^{(\alpha)}(x) \bar{f}^{(\beta)}(y) : \mathrm{d}\mu_{\alpha\beta}(x, y), \tag{2.11}$$

where $\{\mathrm{d}\mu_{\alpha\beta}\}$ is some 2×2 matrix of measures, and

$$: f^\alpha \bar{f}^\beta := f^\alpha \bar{f}^\beta - \langle 0, 0 | f^\alpha \bar{f}^\beta | 0, 0 \rangle. \tag{2.12}$$

Then the expectation value

$$\tau_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \langle N + n, -N - m | \mathrm{e}^{H(\mathbf{t})} g \, \mathrm{e}^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle, \quad N, n, m \in \mathbf{Z}, \tag{2.13}$$

if it exists, is called the τ-function of the two-component Toda lattice (TL) hierarchy or, equivalently, the coupled two-component KP hierarchy (see [6, 26]).

The sets of parameters $\mathbf{t}, \bar{\mathbf{t}}$ and also the integers N, n and m are called 'higher times' of the two-component TL hierarchy. The parameters $\mathbf{t} = (\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ are higher times of the

two-component KP hierarchy, and the second set $\bar{\mathbf{t}} = (\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$ is also a set of higher times, for a different two-component KP hierarchy. Together, they may be referred to as the coupled (two-component) KP hierarchy. The τ-function (2.13) solves an infinite number of bilinear (Hirota) equations. (Further details may be found in [6, 15, 16]. We only note here that each equation contains a certain number of derivatives of the τ-function with respect to the variables $t_k^{(\alpha)}$ and $\bar{t}_k^{(\alpha)}$, $\alpha = 1, 2, k = 1, 2, \ldots$, and the equations may also relate τ-functions with different values of the variables N, n, m.)

To relate $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ to the τ-function in the above sense, we choose the measures $\mathrm{d}\mu_{11}$, $\mathrm{d}\mu_{22}$ and $\mathrm{d}\mu_{21}$ to all vanish, and $\mathrm{d}\mu_{12} := \mathrm{d}\mu$. Thus $g = \mathrm{e}^A$ is of the form (2.5) where A is defined by (2.1). (In this case $: A :$ just coincides with A.) We now will prove that, for $N \geqslant 0$, the resulting τ-function, up to a simple explicit multiplicative factor, is equal to $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$, where the measure $\mathrm{d}\mu$ is the deformed one in (1.3):

$$\tau_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) := \langle N + n, -N - m | \, \mathrm{e}^{H(\mathbf{t})} \, \mathrm{e}^A \, \mathrm{e}^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle \tag{2.14}$$

$$= \frac{1}{N!} (-1)^{\frac{1}{2}N(N+1)+mN} c(\mathbf{t}, \bar{\mathbf{t}}) Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) \tag{2.15}$$

$$= \frac{1}{N!} (-1)^{\frac{1}{2}N(N+1)+mN} c(\mathbf{t}, \bar{\mathbf{t}}) \int \ldots \int \Delta_N(x) \Delta_N(y) \prod_{k=1}^{N} \mathrm{d}\mu(x_k, y_k | \mathbf{t}, n, m, \bar{\mathbf{t}}), \tag{2.16}$$

where

$$c(\mathbf{t}, \bar{\mathbf{t}}) := \exp \left(-\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right). \tag{2.17}$$

Proof. Equations (B.11) and (B.12)) of appendix B imply that

$$\mathrm{e}^{H(\mathbf{t})} f^{(1)}(x) \bar{f}^{(2)}(y) \, \mathrm{e}^{-H(\mathbf{t})} = \mathrm{e}^{V(x, \mathbf{t}^{(1)}) + V(y, \mathbf{t}^{(2)})} f^{(1)}(x) \bar{f}^{(2)}(y), \tag{2.18}$$

$$\mathrm{e}^{-\bar{H}(\bar{\mathbf{t}})} f^{(1)}(x) \bar{f}^{(2)}(y) \, \mathrm{e}^{\bar{H}(\bar{\mathbf{t}})} = \mathrm{e}^{V(x^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y^{-1}, \bar{\mathbf{t}}^{(2)})} f^{(1)}(x) \bar{f}^{(2)}(y). \tag{2.19}$$

Using definition (2.10) one can check that

$$\mathrm{e}^{H(\mathbf{t})} | n, m \rangle = | n, m \rangle, \qquad \langle n, m | \, \mathrm{e}^{H(\bar{\mathbf{t}})} = \langle n, m |, \quad n, m \in \mathbf{Z} \tag{2.20}$$

and, from the Heisenberg algebra relations

$$\left[H_k^{(\alpha)}, H_l^{(\beta)} \right] = k \delta_{\alpha, \beta} \delta_{k, -l}, \quad k, l = \pm 1, \pm 2, \ldots, \quad \alpha, \beta = 1, 2, \tag{2.21}$$

it follows that

$$\mathrm{e}^{H(\mathbf{t})} \, \mathrm{e}^{H(\bar{\mathbf{t}})} = \exp \left(-\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right) \mathrm{e}^{H(\bar{\mathbf{t}})} \, \mathrm{e}^{H(\mathbf{t})}. \tag{2.22}$$

Combining these relations, we have

$$\langle N + n, -N - m | \, \mathrm{e}^{H(\mathbf{t})} g \, \mathrm{e}^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle = \exp \left(-\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right)$$

$$\times \langle N + n, -N - m | \, \mathrm{e}^{A(\mathbf{t}, 0, 0, \bar{\mathbf{t}})} | n, -m \rangle, \tag{2.23}$$

where

$$A(\mathbf{t}, n, m, \bar{\mathbf{t}}) := \int f^{(1)}(x) \bar{f}^{(2)}(y) \, \mathrm{d}\mu(x, y | \mathbf{t}, n, m, \bar{\mathbf{t}}). \tag{2.24}$$

Finally, to remove the numbers n and m from the vacuum states we use (1.55), giving

$$\langle N + n, -N - m | \, \mathrm{e}^{H(\mathbf{t})} g \, \mathrm{e}^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle = \exp\left(- \sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right) (-1)^{mN}$$

$$\times \langle N, -N | \, \mathrm{e}^{A(\mathbf{t},n,m,\bar{\mathbf{t}})} | 0, 0 \rangle \tag{2.25}$$

Now (2.15) follows from (2.23) in the same way that (2.7) was shown equivalent to (2.2). $\square$

2.3. Perturbation series in the variables $\mathbf{t}^{(\alpha)}, \bar{\mathbf{t}}^{(\alpha)}$

In this section we derive the expansions

$$Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = N! \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu) \leqslant N}} g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) \tag{2.26}$$

$$= N! \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu) \leqslant N}} g_{\lambda\mu}^{--}(N, n, m, \mathbf{t}^{(1)}, \mathbf{t}^{(2)}) s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}) \tag{2.27}$$

$$= N! \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu) \leqslant N}} g_{\lambda\mu}^{+-}(N, n, m, \bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}) \tag{2.28}$$

$$= N! \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu) \leqslant N}} g_{\lambda\mu}^{-+}(N, n, m, \mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)}) s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{2.29}$$

where the sums range over all pairs of partitions $\lambda = (\lambda_1, \ldots, \lambda_N), \mu = (\mu_1, \ldots, \mu_N)$ of lengths $\leqslant N$. For such partitions, we also define the labels

$$h_i = \lambda_i - i + N, \qquad h_i' = \mu_i - i + N. \tag{2.30}$$

In terms of these quantities, define four $N \times N$ determinants

$$g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) := \det\left(B_{n+h_i, m+h_j'}(0, 0, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) \right)\big|_{i,j=1,\ldots,N}, \tag{2.31}$$

$$g_{\lambda\mu}^{--}(N, n, m, \mathbf{t}^{(1)}, \mathbf{t}^{(2)}) := \det\left(B_{N+n-h_i-1, N+m+1-h_j'}(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, 0, 0) \right)\big|_{i,j=1,\ldots,N}, \tag{2.32}$$

$$g_{\lambda\mu}^{+-}(N, n, m, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}) := c \det\left(B_{n+h_i, N+m+1-h_j'}(0, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, 0) \right)\big|_{i,j=1,\ldots,N}, \tag{2.33}$$

$$g_{\lambda\mu}^{-+}(N, n, m, \mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)}) := c \det\left(B_{N+n-h_i-1, m+h_j'}(\mathbf{t}^{(1)}, 0, 0, \bar{\mathbf{t}}^{(2)}) \right)\big|_{i,j=1,\ldots,N}, \tag{2.34}$$

where $c := (-1)^{\frac{1}{2}N(N-1)}$, and

$$B_{i,k}(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) := \int x^i y^k \exp(V(x, \mathbf{t}^{(1)}) + V(y, \mathbf{t}^{(2)}) + V(x^{-1}, \bar{\mathbf{t}}^{(1)})$$

$$+ V(y^{-1}, \bar{\mathbf{t}}^{(2)})) \, \mathrm{d}\mu(x, y), \quad i, k \in \mathbf{N}, \tag{2.35}$$

is the matrix of deformed bimoments.

The series (2.26)–(2.29) show that we actually have a 'quadruple' system of one-component TL hierarchies, since each of the double Schur series (2.26)–(2.29) has the form of a Takasaki expansion [24, 25] of a one-component TL τ-function, where the sets of higher time variables are, respectively, $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$, $(\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$, $(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)})$ and $(\bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)})$. One can also view $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ as determining a quadruple system of KP τ-functions, in which each set, $\mathbf{t}^{(1)}, \mathbf{t}^{(2)}, \bar{\mathbf{t}}^{(1)}$ and $\bar{\mathbf{t}}^{(2)}$ plays the role of (one-component) KP higher times.

We begin by proving (2.26). First consider the case $n = m = 0$ and $\bar{\mathbf{t}}^{(1)} = \bar{\mathbf{t}}^{(2)} = 0$, then

$$Z_N(\mathbf{t}, 0, 0, \bar{\mathbf{0}}) = (-1)^{\frac{1}{2}N(N+1)} \langle N, -N| \, \mathrm{e}^{H(\mathbf{t})} A^N |0, 0\rangle. \tag{2.36}$$

Rewrite

$$A = \int f^{(1)}(x) \bar{f}^{(2)}(y) \, \mathrm{d}\mu(x, y) \tag{2.37}$$

in terms of the component operators as

$$A = \sum_{i,k \in \mathbf{Z}} f_i^{(1)} \bar{f}_{-k-1}^{(2)} B_{ik}, \tag{2.38}$$

where

$$B_{ik} = \int x^i y^k \, \mathrm{d}\mu(x, y) \tag{2.39}$$

are the bimoments.

Because of the anticommutation relations

$$\left[f_j^{(1)}, \bar{f}_k^{(2)} \right]_+ = 0, \quad \forall j, k \in \mathbf{Z} \tag{2.40}$$

and

$$f_{-k-1}^{(1)} |0, 0\rangle = 0, \qquad \bar{f}_k^{(2)} |0, 0\rangle = 0, \quad k \geqslant 0, \tag{2.41}$$

nothing is changed by making the substitution

$$A \to A_{++} := \sum_{i,k \geqslant 0} f_i^{(1)} \bar{f}_{-k-1}^{(2)} B_{ik}, \tag{2.42}$$

('projection to the creative corner') when evaluating the τ-function. To compute

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)}) - H^{(2)}(\mathbf{t}^{(2)})} (A_{++})^N |0, 0\rangle, \tag{2.43}$$

we use the relation

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)}) - H^{(2)}(\mathbf{t}^{(2)})} f_{h_1}^{(1)} \bar{f}_{-h_1'-1}^{(2)} \cdots f_{h_N}^{(1)} \bar{f}_{-h_N'-1}^{(2)} |0, 0\rangle = (-1)^{\frac{1}{2}N(N+1)} s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{2.44}$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $\mu = (\mu_1, \ldots, \mu_N)$ are the partitions related to the labels $\{h_i, h_j'\}$ by (2.30), which is proved in appendix B using Wick's theorem.

Note that each term (2.44), for any given pair of decreasing sets of non-negative integers

$$h_1 > h_2 \cdots > h_N \geqslant 0, \qquad h_1' > h_2' \cdots > h_N' \geqslant 0 \tag{2.45}$$

occurs in (2.43) $(N!)^2$ times, multiplied by a product of the moments, with the sign determined by the permutations of the ordering,

$$(-1)^{\mathrm{sgn}(\sigma)} (-1)^{\mathrm{sgn}(\tilde{\sigma})} B_{h_\sigma(1), \tilde{\sigma}(h_1')} \cdots B_{h_\sigma(N), \tilde{\sigma}(h_N')}, \tag{2.46}$$

where $\sigma, \tilde{\sigma} \in S_N$ are the permutations of the indices. (The index sets consist of distinct elements, since all VEVs with repeated values of h_i or h_i' vanish, and re-ordering distinct sets to satisfy (2.45) changes the terms by the sign of the permutation.) Substitution of (2.44) into (2.43) thus gives

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)}) - H^{(2)}(\mathbf{t}^{(2)})} A^N |0, 0\rangle = N! \sum_{\substack{\lambda, \mu \\ \ell(\lambda), \ell\mu) \leqslant N}} g_{\lambda\mu}^{++}(N) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{2.47}$$

where

$$g_{\lambda\mu}^{++}(N) = \det(B_{h_i, h_j'}), \quad i, j = 1, \ldots, N, \tag{2.48}$$

for partitions with $\ell(\lambda), \ell(\mu) \leqslant N$. Finally, we restore the dependence on the remaining variables (n, m, and higher times $(\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})$) by including it in the measure. We thus obtain

$$\langle N+n, -N-m| \, \mathrm{e}^{H(\mathbf{t})} A^N(\bar{\mathbf{t}})|n, -m\rangle = N! \sum_{\substack{\lambda, \mu \\ \ell(\lambda), \ell(\mu) \leqslant N}} g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \quad (2.49)$$

where $A(\bar{\mathbf{t}}) = A(\mathbf{0}, 0, 0, \bar{\mathbf{t}})$ (see (2.24)) and where

$$g_{\lambda\mu}^{++}(N, n, m, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}) = \det\big(B_{h_i+n, h_j'+m}(\bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)})\big), \quad i, j = 1, \ldots, N, \quad (2.50)$$

if $\ell(\lambda), \ell(\mu) \leqslant N$.

Relations (2.27)–(2.29) are proved similarly. First, we equate the variables n, m and all irrelevant higher times to zero. (For the cases (2.27), (2.28) and (2.29), these times are $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$, $(\bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)})$ and $(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)})$, respectively.) Then, for the cases (2.27), (2.28) and (2.29), we consider

$$A \to A_{--} := \sum_{i,k \geqslant 0} f_{N-i-1}^{(1)} \bar{f}_{k-N}^{(2)} B_{N-i-1, 1-k+N}, \quad (2.51)$$

$$A \to A_{+-} := \sum_{i \geqslant, k \geqslant 0} f_{N+i}^{(1)} \bar{f}_{k-N}^{(2)} B_{N+i, 1-k+N}, \quad (2.52)$$

$$A \to A_{-+} := \sum_{i \geqslant, k \geqslant 0} f_{N-i-1}^{(1)} \bar{f}_{-k-1-N}^{(2)} B_{N-i-1, k+N} \quad (2.53)$$

and evaluate

$$\langle N, -N|(A_{--})^N \, \mathrm{e}^{H^{(2)}(\bar{\mathbf{t}}^{(2)}) - H^{(1)}(\bar{\mathbf{t}}^{(1)})}|0, 0\rangle, \quad (2.54)$$

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)})}(A_{+-})^N \, \mathrm{e}^{H^{(2)}(\bar{\mathbf{t}}^{(2)})}|0, 0\rangle, \quad (2.55)$$

$$\langle N, -N| \, \mathrm{e}^{-H^{(2)}(\mathbf{t}^{(2)})}(A_{-+})^N \, \mathrm{e}^{-H^{(1)}(\bar{\mathbf{t}}^{(1)})}|0, 0\rangle. \quad (2.56)$$

For the evaluation of (2.54), (2.55) and (2.56) we use the following formulae (see appendix B):

$$\langle N, -N| f_{N-h_1-1}^{(1)} \bar{f}_{h_1'-N}^{(2)} \cdots f_{N-h_N-1}^{(1)} \bar{f}_{h_N'-N}^{(2)} \, \mathrm{e}^{H^{(2)}(\bar{\mathbf{t}}^{(2)}) - H^{(1)}(\bar{\mathbf{t}}^{(1)})}|0, 0\rangle$$
$$= (-1)^{\frac{1}{2}N(N+1)} s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}), \quad (2.57)$$

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)})} f_{h_1}^{(1)} \bar{f}_{h_1'-N}^{(2)} \cdots f_{h_N}^{(1)} \bar{f}_{h_N'-N}^{(2)} \, \mathrm{e}^{H^{(2)}(\bar{\mathbf{t}}^{(2)})}|0, 0\rangle$$
$$= (-1)^N s_\lambda(\mathbf{t}^{(1)}) s_\mu(\bar{\mathbf{t}}^{(2)}), \quad (2.58)$$

$$\langle N, -N| \, \mathrm{e}^{-H^{(2)}(\mathbf{t}^{(2)})} f_{N-h_1-1}^{(1)} \bar{f}_{-h_1'-1}^{(2)} \cdots f_{N-h_N-1}^{(1)} \bar{f}_{-h_N'-1}^{(1)} \, \mathrm{e}^{-H^{(1)}(\bar{\mathbf{t}}^{(1)})}|0, 0\rangle$$
$$= (-1)^N s_\lambda(\bar{\mathbf{t}}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \quad (2.59)$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ and $\mu = (\mu_1, \ldots, \mu_N)$ are again the partitions related to the labels $\{h_i, h_j'\}$ by (2.30).

Finally, we restore the dependence on the remaining variables ((n, m) and the appropriate higher times, which are respectively $(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$, $(\bar{\mathbf{t}}^{(1)}, \mathbf{t}^{(2)})$ and $(\mathbf{t}^{(1)}, \bar{\mathbf{t}}^{(2)})$ by including these in the definition of the measure, to obtain (2.27), (2.28) and (2.29).

2.4. Generalized two-matrix models with polynomial potentials

In the case of the standard exponentially coupled hermitian two-matrix model [14], and a pair of polynomial *potentials* of degrees $p + 1, q + 1$, or the generalized case of pairs of normal matrices (2NMM) with spectra supported on some specified curve segments (see [3, 4]), we may consider the unperturbed measure as corresponding to the leading monomial potentials,

$$\mathrm{d}\mu(x, y) := \exp\left(-\frac{u_{p+1}}{p+1}x^{p+1} - \frac{u_{q+1}}{q+1}y^{q+1} + xy\right)\mathrm{d}x\,\mathrm{d}y, \tag{2.60}$$

$$\int (*)\,\mathrm{d}\mu(x, y) = \sum_{a=1}^{p+1}\sum_{b=1}^{q+1}\kappa_{ab}\int_{\gamma_a}\int_{\Gamma_b}(*)\exp\left(-\frac{u_{p+1}}{p+1}x^{p+1} - \frac{v_{q+1}}{q+1}y^{q+1} + xy\right)\mathrm{d}x\,\mathrm{d}y, \tag{2.61}$$

with the integration contours $\{\gamma_a, \Gamma_b\}$ chosen in such a way that all the bimoment integrals are finite,

$$B_{jk} := \int x^j y^k \,\mathrm{d}\mu(x, y) = \sum_{i=1}^{p+1}\sum_{j=1}^{q+1}\kappa_{ab}\int_{\gamma_a}\int_{\Gamma_b}x^j y^k$$

$$\times \exp\left(-\frac{u_{p+1}}{p+1}x^{p+1} - \frac{v_{q+1}}{q+1}y^{q+1} + xy\right)\mathrm{d}x\,\mathrm{d}y \leqslant \infty. \tag{2.62}$$

Using the notations of [3, 4], the deformations of the measure may be chosen to be exponentials of lower degree polynomials, defined by parameters $u_1, \ldots, u_p$ and $v_1, \ldots, v_q$, as well as multiplicative monomials in x and y of degrees $n, m \in \mathbb{N}$

$$\mathrm{d}\mu(x, y|\mathbf{u}, \mathbf{v}, n, m) := (-1)^m x^n y^m \exp\left(-\sum_{i=1}^{p}\frac{u_i}{i}x^i - \sum_{i=1}^{q}\frac{v_i}{i}y^i\right)$$

$$\times \exp\left(-\frac{u_{p+1}}{p+1}x^{p+1} - \frac{u_{q+1}}{q+1}y^{q+1} + xy\right)\mathrm{d}x\,\mathrm{d}y. \tag{2.63}$$

Formula (2.26) is applicable in this case if we put $\bar{\mathbf{t}}^{(1)} = \bar{\mathbf{t}}^{(2)} = 0$ and define $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(1)}$ as

$$\mathbf{t}^{(1)} = \mathbf{t_u} := \left(-u_1, -\frac{u_2}{2}, \ldots, -\frac{u_p}{p}, 0, 0, 0, \ldots\right), \tag{2.64}$$

$$\mathbf{t}^{(2)} = \mathbf{t_v} := \left(-v_1, -\frac{v_2}{2}, \ldots, -\frac{v_q}{q}, 0, 0, 0, \ldots\right). \tag{2.65}$$

The partition function may therefore be expressed as

$$Z_N^{2NMM}(\mathbf{u}, n, m, \mathbf{v}) = N!\sum_{\substack{\lambda, \mu \\ \ell(\lambda),\ell(\mu)\leqslant N}} g_{\lambda\mu}(N, n, m)s_\lambda(\mathbf{t_u})s_\mu(\mathbf{t_v}), \tag{2.66}$$

where

$$g_{\lambda\mu}(N, n, m) = \det\left(B_{h_i+n,h'_j+m}\right)\big|_{i,j=1,\ldots,N}, \qquad h_i = \lambda_i - i + N, \qquad h'_i = \mu_i - i + N, \tag{2.67}$$

and B_{jk} are given by (2.62).

3. Direct derivation of Schur function expansions

We now give an alternative derivation of formulae (1.5)–(1.9) through a direct reduction of the multiple double integral to determinants in the bimoments, using the well-known identity [1]

$$\int \prod_{a=1}^{N} \mathrm{d}\mu(x_a, y_a) \det \phi_i(x_j) \det \psi_k(y_l), = N! \det G, \qquad 1 \leqslant i, j, k, l \leqslant N, \tag{3.1}$$

where $\{\phi_i, \psi_i\}_{i=1,\dots,N}$ are an arbitrary set of pairs of functions whose products are integrable with respect to the two variable measure $\mathrm{d}\mu(x, y)$ on some suitable domain and

$$G_{ij} := \int \mathrm{d}\mu(x, y) \phi_i(x) \psi_j(y), \qquad 1 \leqslant i, j \leqslant N. \tag{3.2}$$

We also make use of the following form of the Cauchy–Littlewood identity [19]

$$\exp\left(\sum_{k=1}^{\infty} k t_k \tilde{t}_k\right) = \sum_{\lambda} s_\lambda(\mathbf{t}) s_\lambda(\tilde{\mathbf{t}}), \tag{3.3}$$

where the sum is over all partitions λ.

For any finite set of variables $(x_1, \dots, x_N)$, we denote by $[x]$ the infinite sequence of monomial sums,

$$
\begin{aligned}
[x] &:= \left(\sum_{a=1}^{N} x_a, \frac{1}{2} \sum_{a=1}^{N} x_a^2, \dots \frac{1}{j} \sum_{a=1}^{N} x_a^j, \dots\right), \\
[x^{-1}] &:= \left(\sum_{a=1}^{N} x_a^{-1}, \frac{1}{2} \sum_{a=1}^{N} x_a^{-2}, \dots \frac{1}{j} \sum_{a=1}^{N} x_a^{-j}, \dots\right).
\end{aligned}
\tag{3.4}
$$

The identity (3.3) may be used to express the exponential factors $\mathrm{e}^{V(x, \mathbf{t}^{(1)})}, \mathrm{e}^{V(y, \mathbf{t}^{(2)})}$, $\mathrm{e}^{V(x^{-1}, \bar{\mathbf{t}})^{(1)}}, \mathrm{e}^{V(y^{-1}, \bar{\mathbf{t}}^{(2)})}$ as series in Schur functions:

$$\mathrm{e}^{V(x, \mathbf{t}^{(1)})} = \sum_{\lambda} s_\lambda([x]) s_\lambda(\mathbf{t}^{(1)}), \qquad \mathrm{e}^{V(y, \mathbf{t}^{(2)})} = \sum_{\lambda} s_\lambda([x]) s_\lambda(\mathbf{t}^{(2)}) \tag{3.5}$$

$$\mathrm{e}^{V(x^{-1}, \bar{\mathbf{t}}^{(1)})} = \sum_{\lambda} s_\lambda([x^{-1}]) s_\lambda(\bar{\mathbf{t}}^{(1)}), \qquad \mathrm{e}^{V(y^{-1}, \bar{\mathbf{t}}^{(2)})} = \sum_{\lambda} s_\lambda([x^{-1}]) s_\lambda(\bar{\mathbf{t}}^{(2)}), \tag{3.6}$$

where the sums are over partitions λ with length $\ell(\lambda) \leqslant N$.

Substituting the expansions (3.5) into the matrix integral defining $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ gives

$$
\begin{aligned}
Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) &:= \prod_{a=1}^{N} \left(\int \mathrm{d}\mu(x_a, y_a | \mathbf{t}, n, m, \bar{\mathbf{t}})\right) \Delta_N(x) \Delta_N(y) \\
&= \prod_{a=1}^{N} \left(\int \mathrm{d}\mu(x_a, y_a | \mathbf{0}, n, m, \bar{\mathbf{t}})\right) \Delta_N(x) \Delta_N(y) \\
&\quad \times \sum_{\lambda} s_\lambda(\mathbf{t}^{(1)}) s_\lambda([x]) \sum_{\mu} s_\mu(\mathbf{t}^{(2)}) s_\mu([y]).
\end{aligned}
\tag{3.7}
$$

Making use of the Jacobi–Trudy formula

$$s_\lambda([x]) = \frac{\det\left(x_i^{\lambda_j - j + N}\right)}{\Delta_N(x)}, \qquad s_\mu([x]) = \frac{\det\left(y_k^{\lambda_l - l + N}\right)}{\Delta_N(x)}, \tag{3.8}$$

where

$$\lambda := \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{l(\lambda)}, \qquad \mu := \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{l(\mu)}, \tag{3.9}$$

gives

$$\mathbf{Z}_N = \sum_\lambda \sum_\mu I^{nm}_{\lambda\mu}(\bar{\mathbf{t}}) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{3.10}$$

where

$$I^{nm}_{\lambda\mu}(\bar{\mathbf{t}}) := \prod_{a=1}^N \left(\int \mathrm{d}\mu(x_a, y_a | \mathbf{0}, n, m, \bar{\mathbf{t}}) \right) \det\left(x_i^{\lambda_j - j + N}\right) \det\left(y_k^{\mu_l - l + N}\right). \tag{3.11}$$

Applying the identity (3.1) for $\psi_i(x) := x^i$, $\phi_k(y) := y^k$ then gives

$$I^{nm}_{\lambda\mu} = N! \det G^{nm}_{\lambda\mu}, \tag{3.12}$$

where

$$\left(G^{nm}_{\lambda\mu}\right)_{ij} := \int \mathrm{d}\mu(x, y | \mathbf{0}, n, m, \bar{\mathbf{t}}) x^{\lambda_i - i + N} y^{\mu_j - j + N} = B_{n+h_i, m+h'_j}(0, 0, \bar{\mathbf{t}}^{(1)}, \bar{\mathbf{t}}^{(2)}), \tag{3.13}$$

proving the equality (2.26). The other three cases of equations (2.27)–(2.29) are derived similarly.

We now derive the quadruple Schur function expansion (1.5). To do this, we use the product formula for characters (see [19])

$$s_\lambda([x]) s_\mu([x]) = \sum_\alpha c^\alpha_{\lambda\mu} s_\alpha([x]), \tag{3.14}$$

where the the Littlewood–Richardson coefficients $\{c^\alpha_{\lambda\mu}\}$ are the multiplicities with which the tensor representations of type α appear in the tensor product of those of types λ and μ. We have, as a consequence of the Jacobi–Trudy formula,

$$\begin{aligned} s_\nu([x^{-1}]) &= (x_1 \cdots x_N)^{-\nu_1} s_{\tilde{\nu}}([x]), \\ s_\eta([y^{-1}]) &= (y_1 \cdots y_N)^{-\mu_1} s_{\tilde{\eta}}([y]), \end{aligned} \tag{3.15}$$

where $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_N)$ and $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ and

$$\tilde{\nu}_i := \nu_1 - \nu_{N-i+1}, \qquad \tilde{\eta}_i := \mu_1 - \eta_{N-i+1}. \tag{3.16}$$

Combining this with

$$s_\lambda([x]) s_{\tilde{\nu}}([x]) = \sum_\alpha c^\alpha_{\lambda\tilde{\nu}} s_\alpha([x]), \qquad s_\mu([y]) s_{\tilde{\eta}}([y]) = \sum_\beta c^\beta_{\mu\tilde{\eta}} s_\beta([y]) \tag{3.17}$$

gives

$$Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \sum_{\substack{\lambda, \mu, \nu, \eta \\ \ell(\lambda), \ell(\mu), \ell(\nu), \ell(\eta) \leqslant N}} I_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(\bar{\mathbf{t}}^{(2)}), \tag{3.18}$$

where

$$I_{\lambda\mu\nu\eta}(N, n, m) := \sum_{\alpha, \beta} c^\alpha_{\lambda\tilde{\nu}} c^\beta_{\mu\tilde{\eta}} J^{(n,m)}_{\alpha\beta} \tag{3.19}$$

and

$$J^{(n,m)}_{\alpha\beta} := N! \det(B_{l_i+n, l'_j+m})|_{1 \leqslant i, j \leqslant N}, \tag{3.20}$$

$$l_i := \alpha_i - i + N - \nu_1, \qquad l'_i := \beta_i - i + N - \eta_1. \tag{3.21}$$

4. Summary and further related work

We have given a new two component fermionic representation for $2N$-fold integrals of type (1.1) and used it both to show their relationship to the two-component TL hierarchy and to get double and quadruple Schur function expansions as perturbation series in the deformation parameters, generalizing analogous formulae earlier obtained in [9, 10, 23]. These results were also derived through a 'direct' method, based on standard determinantal identities and character formulae. In another work [11] we show how to use this fermionic representation to reduce the evaluation of multiple integrals of rational symmetric functions arising as determinantal correlation functions in matrix models, using Wick's theorem, to determinantal expressions involving at most double integrals, and the associated biorthogonal polynomials. These results may also alternatively be derived through direct methods [12], based on standard determinantal identities and partial fraction expansions. In [13] a fermionic representation for models of matrices coupled in a chain [7] is also derived.

Acknowledgments

Appendix A. Relation to unitary, hermitian and normal matrix models

In this appendix, we give a number of examples of matrix integrals that reduce to $2N$-fold integrals over their eigenvalues of the form (1.1). Two-matrix models involve integrals of the type

$$\langle F \rangle = \frac{1}{Z_N} \int F(M_1, M_2) \, \mathrm{d}\Omega(M_1, M_2), \qquad Z_N = \int \mathrm{d}\Omega(M_1, M_2), \quad \text{(A.1)}$$

where F is some function of the entries of the matrices M_1 and M_2 and $\mathrm{d}\Omega(M_1, M_2)$ is some measure (possibly complex) on the set of such pairs of matrices. From among the various models that have been studied, we review three types here: (A) normal matrix models, with spectrum supported in open regions of the complex plane; (B) models of pairs of hermitian matrices or, more generally normal matrices with spectrum supported on curve segments in the complex plane and (C) models of unitary matrices. The problem is to reduce the $\sim N^2$ integrations over the independent matrix entries to just $2N$ integrations over the eigenvalues of M_1 and M_2. This is possible for certain choices of matrix measures.

   (A) The simplest are models of normal $N \times N$ matrices M, where

$$M_2^+ = M_1 = M, \qquad [M, M^+] = 0, \quad \text{(A.2)}$$

and the integrand is invariant under $\mathbf{U}(N)$ conjugations. One can then diagonalize the matrix using such transformations,

$$M \rightarrow U M U^\dagger, \qquad U \in \mathbf{U}(N) \quad \text{(A.3)}$$

and integrate over the group $\mathbf{U}(N)$ to reduce the integral to a multiple integral over the eigenvalues with a suitable domain of integration in the complex plane, and the induced

product measure multiplied by the factor $|\Delta_N(z)|^2$. The resulting reduced measure (1.3) can be a rather general one in the complex plane.

For illustrative purposes, we consider the special case when the undeformed measure depends only on the combination $MM^\dagger$, and the deformed partition function is of the form

$$Z_N = \int dM \int dM^\dagger (\det M)^n (\det M^\dagger)^m \exp\left(\mathrm{Tr}(\mathcal{V}(MM^\dagger) + \sum_{i=1}^\infty \left(t_i^{(1)} M^i + t_i^{(2)} (M^\dagger)^i \right) \right).$$

(A.4)

Diagonalizing by $\mathbf{U}(N)$ conjugation and integrating over the group $\mathbf{U}(N)$, this reduces, up to a proportionality constant C_N, to:

$$Z_N = C_N \int dz_1 \, d\bar{z}_1 \cdots \int dz_N \, d\bar{z}_N |\Delta_N(z)|^2 \prod_{a=1}^N z_a^n \bar{z}_a^m \, e^{\mathcal{V}(|z_a|^2) + \sum_{i=1}^\infty (t_i^{(1)} z_a^i + t_i^{(2)} \bar{z}_a^i)}.$$

(A.5)

Because of the polar rotational invariance in each complex z_a-plane, the angular parts of the bimoment integrals appearing in (3.13) may be evaluated. Assuming, e.g. that $n \geqslant m$, the only nonvanishing terms in the sum (3.10) are those for which the partitions λ and μ are related by

$$\lambda_i + n = \mu_i + m.$$

(A.6)

Evaluating the angular parts of the bimoment integrals, (3.10) reduces to

$$Z_N = C_N \sum_{\substack{\lambda \\ \ell(\lambda) \leqslant N}} g_\lambda(n) s_\lambda(\mathbf{t}^{(1)}) s_{\lambda + n - m}(\mathbf{t}^{(2)})$$

(A.7)

where $\lambda + n - m$ denotes the partition $(\lambda_1 + n - m, \dots, \lambda_N + n - m)$ and

$$g_\lambda(n) := \pi^N \prod_{i=1}^N \left(\int_0^\infty e^{\mathcal{V}(X)} X^{\lambda_i - i + n + N} \, dX \right).$$

(A.8)

(B) In the case where (M_1, M_2) are a pair of independent hermitian matrices, or normal matrices with spectral supports along some specified curve segments, the problem is more involved. The first example is the hermitian two-matrix model of Itzykson and Zuber [14] (see also [20], where these are referred to as unitary ensembles). The partition function is

$$I_N = \iint e^{\mathrm{Tr}(V_1(M_1) + V_2(M_2))} e^{\mathrm{Tr} M_1 M_2} \, d\Omega(M_1) \, d\Omega(M_2)$$

(A.9)

where

$$d\Omega(M) = \prod_{i=1}^N dM_{ii} \prod_{i<j}^N d\,\mathrm{Re}\,M_{ij} \prod_{i<j}^N d\,\mathrm{Im}\,M_{ij}.$$

(A.10)

The *potentials* V_1, V_2 can be fairly general, but most often are taken as polynomials. Diagonalizing the matrices M_1 and M_2 via two distinct $\mathbf{U}(N)$ conjugations,

$$M_i = U_i X U_i^{-1}, \quad i = 1, 2, \qquad X := \mathrm{diag}(x_1, \dots, x_N), \qquad Y := \mathrm{diag}(y_1, \dots, y_N),$$

(A.11)

and integrating over U_1, one obtains

$$I_N = V_N \iint dx_1 \, dy_1 \cdots \iint dx_N \, dy_N \, J_N(X, Y) \Delta_N(x)^2 \Delta_N(y)^2 \prod_{i=1}^N e^{\mathrm{Tr}(V_1(x_i) + V_2(y_i))},$$

(A.12)

where

$$J_N(X, Y) := \int_{\mathbf{U}(N)} e^{\mathrm{Tr} UXU^{\dagger}Y} d_* U \tag{A.13}$$

is the remaining integral over the unitary group, $d_* U$ is the normalized Haar measure and

$$V_N := \frac{(2\pi)^{N(N-1)}}{\left(\prod_{k=1}^{N} k!\right)^2}. \tag{A.14}$$

The observation of [14] was that

$$\int_{\mathbf{U}(N)} e^{\mathrm{Tr} UXU^{\dagger}Y} d_* U = \left(\prod_{k=1}^{N-1} k!\right) \frac{\det(e^{x_i y_j})}{\Delta_N(x)\Delta_N(y)}. \tag{A.15}$$

Using the anti-symmetry of determinants and changes of variables, one then obtains the reduced integral

$$I_N = \frac{(2\pi)^{N(N-1)}}{\prod_{k=1}^{N} k!} \iint dx_2 \, dy_1 \ldots \iint dx_N \, dy_N \Delta_N(x)\Delta_N(y) \prod_{i=1}^{N} e^{V_1(x_i)+V_2(y_i)} e^{x_i y_i}, \tag{A.16}$$

which is proportional to (1.1) for measures (1.3) of the form

$$d\mu(x, y) = e^{V_1(x)+V_2(y)} e^{xy} \, dx \, dy. \tag{A.17}$$

This same computation is valid for any family of matrices that are unitarily diagonalizable, even if the spectral support is not on the real axis. Thus, if (M_1, M_2) are taken as normal matrices with spectrum supported on some union of curves $\{\gamma_a\}$, for M_1 and $\{\Gamma_b\}$ for M_2, then the reduced form of the partition function is obtained by replacing the double integrals over the real axes in (A.12) by integrals of the form given, e.g. in equation (2.61) for the case of polynomial potentials [3, 4].

Generalizations of this construction were considered in [22, 23, 28]. We may replace the interaction term $e^{\mathrm{Tr}(M_1 M_2)}$ in (A.9) by a more general one of the form

$$\tau_r(N, M_1 M_2) := \sum_{\substack{\lambda \\ \ell(\lambda) \leqslant N}} d_{\lambda,N} r_{\lambda}(N) s_{\lambda}(M_1 M_2), \tag{A.18}$$

where $\{r(j)\}_{j\in\mathbf{N}^+}$ is some sequence of complex numbers,

$$r_{\lambda}(N) := \prod_{i,j\in\lambda} r(N + j - i), \tag{A.19}$$

with the product ranging over all nodes of the Young diagram λ with positions (i, j), and

$$d_{\lambda,N} = s_{\lambda}(\mathbf{I}_N) \tag{A.20}$$

is the dimension of the representation of $\mathbf{GL}(N)$ given by irreducible tensors of type λ. (We use here the abbreviated notation $\tau_r(N, M_1 M_2)$ to express what, in the more general setting of [9, 22, 23] was denoted $\tau_r(N, \mathbf{I}_N, M_1 M_2)$. Note that expressions such as (A.18), viewed as functions of the trace invariants of the matrix $M_1 M_2$, are also KP τ-functions.) Here the Schur functions $s_{\lambda}(Z)$ are interpreted as conjugation invariant functions defined on the space of complex $N \times N$ matrices Z (see appendix B). Then $J_N(X, Y)$ is replaced in equation (A.12) by the following more general integral (see [22, 23] for details):

$$J_{N,r}(X, Y) := \int_{\mathbf{U}(N)} \tau_r(N, UXU^{-1}Y) d_* U = \sum_{\substack{\lambda \\ \ell(\lambda) \leqslant N}} r_{\lambda} s_{\lambda}(X) s_{\lambda}(Y) \tag{A.21}$$

$$= C_{N,r} \frac{\det(\tau_r(1, x_i y_j))}{\Delta_N(x) \Delta_N(y)}, \tag{A.22}$$

where

$$\tau_r(1, x) = 1 + \sum_{k=1}^{\infty} r(1) \cdots r(k) x^k, \tag{A.23}$$

and

$$C_{N,r} := \frac{1}{\prod_{k=1}^{N-1} \prod_{j=1}^{k} r(j)}. \tag{A.24}$$

The reduced integral is again of the form (1.1) with the measure (A.17) replaced by

$$d\mu(x, y) = e^{V_1(x) + V_2(y)} \tau_r(1, xy) \, dx \, dy. \tag{A.25}$$

Note that, setting $Y = \mathbf{I}_N$ on the left-hand side of equation (A.22), and taking the limits $\{y_j \to 1\}_{j=1,\dots,N}$ on the right gives

$$\tau_r(N, X) = \frac{C_{N,r}}{\prod_{k=1}^{N-1} k!} \frac{\det(x_i^{N-j} \tau_r^{(N-j)}(1, x_i))}{\Delta_N(x)}. \tag{A.26}$$

Choosing

$$\tau_r(1, x) = e^x, \qquad r(j) = \frac{1}{j}, \tag{A.27}$$

we obtain from (A.26)

$$\tau_r(N, X) = e^{\mathrm{tr}(X)}, \tag{A.28}$$

which is the Itzykson–Zuber case.

Another example is obtained by choosing

$$\tau_r(1, x) = \frac{1}{(1 - zx)^{a-N+1}}, \qquad r(j) = \frac{z(a - N + j)}{j} \tag{A.29}$$

for some pair of constants (a, z). In this case

$$C_{N,r} = \frac{\prod_{k=1}^{N-1} k!}{z^{\frac{1}{2} N(N-1)} \prod_{k=1}^{N-1} (a - N + 1)_k}, \tag{A.30}$$

where

$$(a - N + 1)_k := \prod_{j=1}^{k} (a - N + j) \tag{A.31}$$

is the Pochhammer symbol, and

$$J_{N,r}(X, Y) = C_{N,r} \frac{\det(1 - z x_i y_j)^{N-a-1}}{\Delta_N(x) \Delta_N(y)}. \tag{A.32}$$

Equation (A.26) therefore gives

$$\tau_r(N, X) = \det(\mathbf{I}_N - zX)^{-a}. \tag{A.33}$$

Thus the integral (A.12) is replaced by

$$I_{N,r} = \iint e^{\mathrm{Tr}(V_1(M_1) + V_2(M_2))} \det(\mathbf{I}_N - zM_1 M_2)^{-a} \, d\Omega(M_1) \, d\Omega(M_2)$$

$$= \left(\frac{2\pi^2}{z} \right)^{\frac{1}{2} N(N-1)} \frac{1}{\prod_{k=1}^{N-1} (a - N + 1)_k} \iint \Delta_N(x) \Delta_N(y) \prod_{i=1}^{N} \frac{e^{V_1(x_i) + V_2(y_i)}}{(1 - z x_i y_i)^{a-N+1}} \, dx_i \, dy_i, \tag{A.34}$$

which is proportional to (1.1) for measures (1.3) of the form

$$d\mu(x, y) = e^{V_1(x)+V_2(y)}(1 - zxy)^{N-a-1}\, dx\, dy. \tag{A.35}$$

Remark A.1. For the case $a = 0$ the right-hand side of (A.32), up to the constant factor $C_{N,r}$, is

$$\frac{\det((1 - zx_i y_j)^{N-1})|_{i,j=1,\ldots,N}}{\Delta_N(x)\Delta_N(y)}, \tag{A.36}$$

which itself is a constant in the variables $\{x_i, y_i\}$, as of course it must be, since there is no coupling in this case between the pairs of matrices (M_1, M_2). This may be seen by noting that the numerator and denominator are polynomials of the same degree $N - 1$ in all the variables, and antisymmetric under the interchange of any pair $x_i \leftrightarrow x_j$ or $y_i \leftrightarrow y_j$. This is therefore a rational function without poles, hence a polynomial, which is bounded in all the variables, and therefore a constant.

(C) For unitary two-matrix models [23] (referred to in [20] as circular ensembles), $M_i \in \mathbf{U}(N), i = 1, 2$, we have

$$I_N^{gen} = \int_{\mathbf{U}(N)} \int_{\mathbf{U}(N)} e^{\operatorname{Tr}(V_1(M_1)+V_2(M_2))} \tau_r(N, (M_1 M_2)^{-1}) d_* M_1 d_* M_2$$

$$= C_{N,r} V_N N! \oint \ldots \oint \Delta_N(x)\Delta_N(y) \prod_{i=1}^{N} e^{V_1(x_i)+V_2(y_i)} \tau_r(1, (x_i y_i)^{-1}) \frac{dx_i}{x_i} \frac{dy_i}{y_i}, \tag{A.37}$$

where $d_* M_1, d_* M_2$ are the Haar measure on two copies of the group $\mathbf{U}(N)$, and x_i and $y_i, i = 1, \ldots, N$ are eigenvalues of the matrices M_1 and M_2, with values on the unit circle in the complex x_i and y_i planes. This is related to measures in (1.3) of the form

$$d\mu = e^{V_1(x)+V_2(y)} \tau_r(1, (xy)^{-1}) \frac{dx\, dy}{xy}. \tag{A.38}$$

An example is the case

$$\tau_r(N, (M_1 M_2)^{-1}) = e^{\operatorname{Tr}(M_1 M_2)^{-1}} \tag{A.39}$$

$$\tau_r(1, (xy)^{-1}) = e^{(xy)^{-1}}, \tag{A.40}$$

which was considered in detail in [28, 29], where the large N limit of this model was also studied.

Another simple example, which is the unitary analog of the one above, is

$$I_N^{gen} = \int\int e^{\operatorname{Tr}(V_1(M_1)+V_2(M_2))} \det(\mathbf{I}_N - z(M_1 M_2)^{-1})^{-a} d_* M_1 d_* M_2$$

$$= C_{N,r} V_N N! \oint \ldots \oint \Delta_N(x)\Delta_N(y) \prod_{i=1}^{N} \frac{e^{V_1(x_i)+V_2(y_i)}}{\left(1 - \frac{z}{x_i y_i}\right)^{(a-N+1)}} \frac{dx_i}{x_i} \frac{dy_i}{y_i}, \tag{A.41}$$

which corresponds to measures in (1.3) of the form

$$d\mu = e^{V_1(x)+V_2(y)} \left(1 - \frac{z}{xy}\right)^{N-a-1} \frac{dx\, dy}{xy}. \tag{A.42}$$

In this case, r is the same as (A.29)

$$r(j) := \frac{z(a - N + j)}{j}, \tag{A.43}$$

and hence

$$\tau_r(N, (M_1M_2)^{-1}) = \det(\mathbf{I}_N - z(M_1M_2)^{-1})^{-a} \tag{A.44}$$

$$\tau_r\left(1, \frac{1}{xy}\right) = \frac{1}{\left(1 - \frac{z}{xy}\right)^{a-N+1}}. \tag{A.45}$$

Appendix B. Derivation of formula (2.44)

In this appendix, we recall some definitions about Schur functions (see [19] for further details) and results from [6, 15] that are needed in the derivation.

A *partition* is a sequence of non-negative integers in weakly decreasing order:

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots), \qquad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r \geqslant \ldots, \tag{B.1}$$

where we identify $(\lambda_1, \ldots, \lambda_n)$ with $(\lambda_1, \ldots, \lambda_n, 0)$ and, in the case of an infinite sequence, assume $\lambda_r = 0$ for $r \gg 0$. The λ_i in (B.1) are called the *parts* of λ. The number of nonzero parts is the *length* of λ, denoted by $\ell(\lambda)$. We use the notation $\mathbf{t} = (t_1, t_2, \ldots)$ and $\bar{\mathbf{t}} = (\bar{t}_1, \bar{t}_2, \ldots)$ in what follows. The elementary Schur functions $\{s_i(\mathbf{t})\}_{i\in\mathbf{N}}$ are defined by

$$\exp(V(x, \mathbf{t})) = \sum_{k\geqslant 0} s_i(\mathbf{t})x^k, \tag{B.2}$$

where

$$V(x, \mathbf{t}) := \sum_{k=1}^{\infty} t_k x^k. \tag{B.3}$$

The Schur function $s_\lambda(\mathbf{t})$ corresponding to a partition λ is given by

$$s_\lambda(\mathbf{t}) = \det\left(s_{\lambda_i - i + j}(\mathbf{t})\right)_{1\leqslant i, j\leqslant \ell(\lambda)}, \tag{B.4}$$

where, for $k < 0$, we put $s_k = 0$. When the sequence $\{t_j\}_{j\in\mathbf{N}}$ is identified with the elementary trace invariants $\left\{\frac{1}{j}\mathrm{Tr}(Z^j)\right\}_{j\in\mathbf{N}}$ of elements $Z \in GL(n)$ or $Z \in U(n)$, the Schur function $s_\lambda(\mathbf{t})$ is the trace (character) of the rank $\ell(\lambda)$ irreducible tensor representation whose symmetries are given by the Young diagram associated to the partition λ. When interpreted in this way, as a class function on the group $GL(n)$, its values are denoted $s_\lambda(Z)$, and this may be extended, by continuity, to all complex $N \times N$ matrices Z.

Using the notations of subsections 1.1 and 1.2 we define, for $m \neq 0$,

$$H_m := \sum_{i\in\mathbf{Z}} f_i \bar{f}_{i+m}. \tag{B.5}$$

It follows that

$$[H_m, f_i] = f_{i-m}, \qquad [H_m, \bar{f}_i] = -\bar{f}_{i+m}, \tag{B.6}$$

and also that

$$H_m|0\rangle = 0, \quad m > 0, \tag{B.7}$$

and hence

$$\exp\left(\sum_{m=1}^{\infty} H_m t_m\right)|0\rangle = |0\rangle. \tag{B.8}$$

For any $a \in \mathcal{A}$, let

$$a(\mathbf{t}) := \exp\left(\sum_{m=1}^{\infty} H_m t_m\right) a \exp\left(-\sum_{m=1}^{\infty} H_m t_m\right) = \exp\left(\text{ad} \sum_{m=1}^{\infty} H_m t_m\right) a, \tag{B.9}$$

$$a(\bar{\mathbf{t}}) := \exp\left(\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) a \exp\left(-\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) = \exp\left(\text{ad} \sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) a. \tag{B.10}$$

Using (B.6) it is easily verified that

$$f(x)(\mathbf{t}) = \exp(V(x, \mathbf{t})) f(x), \qquad \bar{f}(y)(\mathbf{t}) = \exp(-V(y, \mathbf{t})) \bar{f}(y), \tag{B.11}$$

$$f(x)(\bar{\mathbf{t}}) = \exp(V(x^{-1}, \bar{\mathbf{t}})) f(x), \qquad \bar{f}(y)(\bar{\mathbf{t}}) = \exp(-V(y^{-1}, \bar{\mathbf{t}})) \bar{f}(y), \tag{B.12}$$

and hence

$$f_i(\mathbf{t}) = \sum_{k \geqslant 0} s_k(\mathbf{t}) f_{i-k}, \qquad \bar{f}_i(\mathbf{t}) = \sum_{k \geqslant 0} s_k(-\mathbf{t}) \bar{f}_{i+k}, \tag{B.13}$$

$$f_i(\bar{\mathbf{t}}) = \sum_{k \geqslant 0} s_k(\bar{\mathbf{t}}) f_{i+k}, \qquad \bar{f}_i(\bar{\mathbf{t}}) = \sum_{k \geqslant 0} s_k(-\bar{\mathbf{t}}) \bar{f}_{i-k}. \tag{B.14}$$

Define the states

$$\langle N| := \langle 0| \bar{f}_0 \cdots \bar{f}_{N-1}, \qquad \langle -N| := \langle 0| f_{-1} \cdots f_{-N}. \tag{B.15}$$

Now consider the integers $h_1 > \cdots > h_N \geqslant 0$ related to a partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ by

$$h_i = \lambda_i - i + N. \tag{B.16}$$

It follows from (B.8) that

$$\langle N| \exp\left(\sum_{m=1}^{\infty} H_m t_m\right) f_{h_1} \cdots f_{h_N} |0\rangle = \langle N| f_{h_1}(\mathbf{t}) \cdots f_{h_N}(\mathbf{t}) |0\rangle. \tag{B.17}$$

By Wick's theorem and equations (B.13) and (B.14) this is equal to

$$\det(\langle 0| \bar{f}_{N-i} f_{h_j}(\mathbf{t}) |0\rangle)|_{i,j=1,\ldots,N} = \det s_{\lambda_j - j + i}(\mathbf{t})|_{i,j=1,\ldots,N} = s_\lambda(\mathbf{t}), \tag{B.18}$$

where the second equality follows from (B.4). Thus we obtain

$$\langle N| \exp\left(\sum_{m=1}^{\infty} H_m t_m\right) f_{h_1} \cdots f_{h_N} |0\rangle = s_\lambda(\mathbf{t}). \tag{B.19}$$

Similarly, we have

$$\langle -N| \exp\left(\sum_{m=1}^{\infty} H_m t_m\right) \bar{f}_{-h_1-1} \cdots \bar{f}_{-h_N-1} |0\rangle = \det \langle 0| f_{i-N-1} \bar{f}_{-h_j-1}(\mathbf{t}) |0\rangle|_{i,j=1,\ldots,N}$$

$$= \det s_{\lambda_j - j + i}(-\mathbf{t})|_{i,j=1,\ldots,N}$$

$$= s_\lambda(-\mathbf{t}). \tag{B.20}$$

To prove equation (2.44) we use factorization, (B.19) and (B.20),

$$\langle N, -N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)}) - H^{(2)}(\mathbf{t}^{(2)})} f_{h_1}^{(1)} \bar{f}_{-h'_1-1}^{(2)} \cdots f_{h_N}^{(1)} \bar{f}_{-h'_N-1}^{(2)} |0, 0\rangle$$

$$= (-1)^{\frac{1}{2} N(N+1)} \langle N| \, \mathrm{e}^{H^{(1)}(\mathbf{t}^{(1)})} f_{h_1}^{(1)} \cdots f_{h_N}^{(1)} |0\rangle \langle -N| \, \mathrm{e}^{-H^{(2)}(\mathbf{t}^{(2)})} \bar{f}_{-h'_1-1}^{(2)} \cdots \bar{f}_{-h'_N-1}^{(2)} |0\rangle \tag{B.21}$$

$$= (-1)^{\frac{1}{2}N(N+1)} s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}), \tag{B.22}$$

where

$$h_i := \lambda_i - i + N, \qquad h_i' := \mu_i - i + N. \tag{B.23}$$

Similarly, from

$$\langle \pm N | H_{-m} = 0 \tag{B.24}$$

it follows that

$$\langle \pm N | \exp\left(\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) = \langle \pm N |. \tag{B.25}$$

We therefore have

$$\langle N | f_{N-h_N-1} \cdots f_{N-h_1-1} \exp\left(-\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) |0\rangle = \langle N | f_{N-h_N-1}(\bar{\mathbf{t}}) \cdots f_{N-h_1-1}(\bar{\mathbf{t}}) |0\rangle \tag{B.26}$$

By Wick's theorem this is equal to

$$\det(\langle 0 | \bar{f}_{i-1} f_{N-h_j-1}(\bar{\mathbf{t}}) |0\rangle)|_{i,j=1,\ldots,N} = \det s_{\lambda_j-j+i}(\bar{\mathbf{t}})|_{i,j=1,\ldots,N} = s_\lambda(\bar{\mathbf{t}}), \tag{B.27}$$

where the last equality follows from (B.14). Thus

$$\langle N | f_{N-h_1-1} \cdots f_{N-h_N-1} \exp\left(-\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) |0\rangle = (-1)^{\frac{1}{2}N(N-1)} s_\lambda(\bar{\mathbf{t}}). \tag{B.28}$$

Similarly, we obtain

$$\langle -N | \bar{f}_{h_N-N} \cdots \bar{f}_{h_1-N} \exp\left(-\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) |0\rangle = \det \langle 0 | f_{-i} \bar{f}_{h_j-N}(\bar{\mathbf{t}}) |0\rangle|_{i,j=1,\ldots,N}$$

$$= \det s_{\lambda_j-j+i}(-\bar{\mathbf{t}})|_{i,j=1,\ldots,N} = s_\lambda(-\bar{\mathbf{t}}), \quad \text{(B.29)}$$

and hence

$$\langle -N | \bar{f}_{h_1-N} \cdots \bar{f}_{h_N-N} \exp\left(\sum_{m=1}^{\infty} H_{-m} \bar{t}_m\right) |0\rangle = (-1)^{\frac{1}{2}N(N-1)} s_\lambda(\bar{\mathbf{t}}). \tag{B.30}$$

By factorization of the appropriate VEV of two-component fermions into the product of VEVs of one component fermions, equations (B.19), (B.20), (B.28) and (B.30) then imply (2.44), as well as (2.57), (2.58) and (2.59).

Remark B.1. Note that all partitions λ considered in this appendix have length $\ell(\lambda) \leqslant N$.

Appendix C. Quadruple Schur function series for τ-functions of two component Toda lattice and derivation of formula (1.5)

Using results from [6, 15] one proves (see e.g. [9]) the formulae

$$\langle N | \exp\left(\sum_{k=1}^{\infty} H_k t_k\right) = \sum_\lambda s_\lambda(\mathbf{t}) \langle N; \lambda |, \qquad \exp\left(\sum_{k=1}^{\infty} H_{-k} t_k\right) |N\rangle = \sum_\lambda s_\lambda(\mathbf{t}) |N; \lambda\rangle,$$

$$\tag{C.1}$$

where the sums range over all partitions and the Fock vectors on the right hand sides are defined as follows

$$\langle N; \lambda | := a(N, \lambda) \langle N | \prod_{i=1}^{k} \bar{f}_{N+\alpha_i} f_{N-\beta_i-1} \tag{C.2}$$

$$|N; \lambda \rangle := a(N, \lambda) \prod_{i=1}^{k} \bar{f}_{N-\beta_i-1} f_{N+\alpha_i} |N \rangle. \tag{C.3}$$

where

$$a(N, \lambda) = (-1)^{\beta_1 + \cdots + \beta_k + \frac{1}{2} N(N-1)}. \tag{C.4}$$

Here k is the number of diagonal nodes of the Young diagram corresponding to λ, α_j is the number of nodes in the jth row to the right of the (j, j) diagonal one and β_j is the number of nodes in the column below it. It follows that the sets of integers $\{\alpha_i\}$, $\{\beta_i\}$ are strictly decreasing $(\alpha_1 > \cdots > \alpha_k)$ and $(\beta_1 > \cdots > \beta_k)$, and they uniquely determine λ. The partition λ is expressed in Frobenius notation (see [19]) as

$$\lambda = (\alpha_1, \ldots, \alpha_k \mid \beta_1, \ldots, \beta_k). \tag{C.5}$$

The transposed partition is denoted λ^{tr}, which in Frobenius notation is

$$\lambda^{\text{tr}} := (\beta_1, \ldots, \beta_k \mid \alpha_1, \ldots, \alpha_k). \tag{C.6}$$

From the Jacobi–Trudy formulae (3.8) it follows that

$$f(x_1) \cdots f(x_N)|0\rangle = \sum_{\substack{\lambda \\ \ell(\lambda) \leqslant N}} |N; \lambda\rangle s_\lambda([x]) \Delta_N(x) \tag{C.7}$$

$$\bar{f}(y_1) \cdots \bar{f}(y_N)|0\rangle = \sum_{\substack{\lambda \\ \ell(\lambda) \leqslant N}} |-N; \lambda\rangle (-1)^{|\lambda|} s_{\lambda^{\text{tr}}}([y]) \Delta_N(y). \tag{C.8}$$

In the two component setting, we use the following notations:

$$\langle N^{(1)}, N^{(2)}; \lambda^{(1)}, \lambda^{(2)} | := a(N^{(1)}, \lambda^{(1)}) a(N^{(2)}, \lambda^{(2)})$$
$$\times \sum_{i=1}^{k^{(2)}} \langle N^{(1)}, N^{(2)} | \prod_{i=1}^{k^{(1)}} \bar{f}^{(1)}_{N^{(1)}+\alpha_i} f^{(1)}_{N^{(1)}-\beta_i-1} \prod_{i=1}^{k^{(2)}} \bar{f}^{(2)}_{N^{(1)}+\alpha_i} f^{(2)}_{N^{(1)}-\beta_i-1}, \tag{C.9}$$

$$|N^{(1)}, N^{(2)}; \lambda^{(1)}, \lambda^{(2)} \rangle := a(N^{(1)}, \lambda^{(1)}) a(N^{(2)}, \lambda^{(2)})$$
$$\times \prod_{i=1}^{k^{(1)}} \bar{f}^{(1)}_{N^{(1)}-\beta_i-1} f^{(1)}_{N^{(1)}+\alpha_i} \prod_{i=1}^{k^{(2)}} \bar{f}^{(2)}_{N^{(2)}-\beta_i-1} f^{(2)}_{N^{(2)}+\alpha_i} |N^{(1)}, N^{(2)} \rangle \tag{C.10}$$

where

$$\lambda^{(j)} = \left(\alpha_1^{(j)}, \ldots, \alpha_{k^{(j)}}^{(j)} \mid \beta_1^{(j)}, \ldots, \beta_{k^{(j)}}^{(j)}\right), \quad j = 1, 2. \tag{C.11}$$

The vectors $\{\langle N^{(1)}, N^{(2)}; \lambda^{(1)}, \lambda^{(2)} |\}$ and $\{|N^{(1)}, N^{(2)}; \lambda^{(1)}, \lambda^{(2)} \rangle\}$ form dual orthonormal bases for the left and right Fock spaces:

$$\langle N^{(1)}, N^{(2)}; \lambda^{(1)}, \lambda^{(2)} | M^{(1)}, M^{(2)}; \mu^{(1)}, \mu^{(2)} \rangle = \delta_{N^{(1)}, M^{(1)}} \delta_{N^{(2)}, M^{(2)}} \delta_{\lambda^{(1)}, \mu^{(1)}} \delta_{\lambda^{(2)}, \mu^{(2)}}. \tag{C.12}$$

Using (C.1) for each component, we then have

$$\langle N^{(1)}, N^{(2)} | \exp \left(\sum_{k=1}^{\infty} H_k^{(1)} t_k^{(1)} - \sum_{k=1}^{\infty} H_k^{(2)} t_k^{(2)} \right) = \sum_{\lambda, \mu} s_\lambda(\mathbf{t}^{(1)}) s_\mu(-\mathbf{t}^{(2)}) \langle N^{(1)}, N^{(2)}; \lambda, \mu |,$$

(C.13)

$$\exp \left(\sum_{k=1}^{\infty} H_{-k}^{(2)} t_k^{(2)} - \sum_{k=1}^{\infty} H_{-k}^{(1)} t_k^{(1)} \right) | N^{(1)}, N^{(2)} \rangle = \sum_{\lambda, \mu} s_\lambda(\mathbf{t}^{(1)}) s_\mu(-\mathbf{t}^{(2)}) | N^{(1)}, N^{(2)}; \lambda, \mu \rangle.$$

(C.14)

This implies that each two-component TL τ-function (2.13) may be expanded in a quadruple Schur function series:

$$\langle N + n, -N - m | \, e^{H(\mathbf{t})} g \, e^{\bar{H}(\bar{\mathbf{t}})} | n, -m \rangle$$
$$= \sum_{\lambda, \mu, \nu, \eta} g_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(-\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(-\bar{\mathbf{t}}^{(2)}),$$

(C.15)

where

$$g_{\lambda\mu\nu\eta}(N, n, m) := \langle N + n, -N - m; \lambda, \mu | g | n, -m; \nu, \eta \rangle$$

(C.16)

Remark C.1. The signs of the arguments of the Schur functions on the right hand side of (C.15) can be reversed using (see [19])

$$s_\lambda(\mathbf{t}) = (-1)^{|\lambda|} s_{\lambda^{\mathrm{tr}}}(-\mathbf{t}).$$

(C.17)

Here $|\lambda|$ denotes weight of λ, which in the Frobenius notation equals $k + \alpha_1 + \cdots + \alpha_k + \beta_1 + \cdots + \beta_k$.

It follows from equation (2.15) that the partition function $Z_N(\mathbf{t}, n, m, \mathbf{t})$ is given by (C.15).

$$Z_N(\mathbf{t}, n, m, \mathbf{t}) = (-1)^{\frac{1}{2}N(N+1)+mN} \exp \left(\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right)$$
$$\times \sum_{\lambda, \mu, \nu, \eta} g_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(-\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(-\bar{\mathbf{t}}^{(2)}),$$

(C.18)

where, by the Cauchy–Littlewood identity

$$\exp \left(\sum_{\alpha=1}^{2} \sum_{k=1}^{\infty} k t_k^{(\alpha)} \bar{t}_k^{(\alpha)} \right) = \sum_{\lambda, \mu, \nu, \eta} s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(\bar{\mathbf{t}}^{(2)}).$$

(C.19)

Using (C.17) and the product formula

$$s_\lambda(\mathbf{t}) s_\mu(\mathbf{t}) = \sum_\alpha c_{\lambda\mu}^\alpha s_\alpha(\mathbf{t}),$$

(C.20)

where $c_{\lambda\mu}^\alpha$ are the Littlewood-Richardson coefficients (see [19]), we obtain

$$Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \sum_{\lambda, \mu, \nu, \eta} I_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(\bar{\mathbf{t}}^{(2)}),$$

(C.21)

where

$$I_{\lambda\mu\nu\eta}(N, n, m) = (-1)^{\frac{1}{2}N(N+1)+mN} N!$$
$$\times \sum_{\substack{\lambda', \mu', \nu', \eta' \\ \lambda'', \mu'', \nu'', \eta''}} (-1)^{|\mu'|+|\eta'|} c_{\lambda'\lambda''}^\lambda c_{\mu'\mu''}^\mu c_{\nu'\nu''}^\nu c_{\eta'\eta''}^\eta g_{\lambda', \mu', \nu', \eta'}(N, n, m).$$

(C.22)

Wick's theorem can then be used to evaluate the right-hand side of (C.16) and express the result in terms of the bimoments. The quadruple Schur expansion follows here from the fact that $Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}})$ is essentially a τ function for the two-component TL hierarchy.

Rather than detailing this calculation, we now give an alternative method of evaluating $I_{\lambda\mu\nu\eta}(N, n, m)$ that uses the particular form of g in equation (2.5), and which directly yields the vanishing of $I_{\lambda\mu\nu\eta}(N, n, m)$ if the length of any of the partitions λ, μ, ν, η exceeds N.

Using (2.15) and (C.13), we have

$$
\begin{aligned}
Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) &= (-1)^{\frac{1}{2}N(N+1)}\langle N, -N| \, \mathrm{e}^{H(\mathbf{t})} A(\mathbf{0}, n, m, \bar{\mathbf{t}})^N |0, 0\rangle \\
&= (-1)^{\frac{1}{2}N(N+1)} \sum_{\lambda,\mu} s_\lambda(\mathbf{t}^{(1)}) s_\mu(-\mathbf{t}^{(2)}) \langle N, -N; \lambda, \mu | A(\mathbf{0}, n, m, \bar{\mathbf{t}})^N |0, 0\rangle,
\end{aligned}
$$
(C.23)

where

$$
A(\mathbf{0}, n, m, \bar{\mathbf{t}})^N = \prod_{i=1}^{N} \int f^{(1)}(x_i) \bar{f}^{(2)}(y_i) x^n (-y)^m \, \mathrm{e}^{V(x^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y^{-1}, \bar{\mathbf{t}}^{(2)})} \, \mathrm{d}\mu(x_i, y_i).
$$
(C.24)

It follows from (C.7) and (C.8), the sign count from interchanging the orders of fermionic operators, and the orthogonality relations that

$$
\langle N, -N; \lambda, \mu | \prod_{i=1}^{N} f^{(1)}(x_i) \bar{f}^{(2)}(y_i) |0, 0\rangle = (-1)^{\frac{1}{2}N(N+1)} s_\lambda([x]) s_{\mu^{\mathrm{tr}}}([y]) \Delta_N(x) \Delta_N(y),
$$
(C.25)

which vanishes unless $\ell(\lambda), \ell(\mu^{\mathrm{tr}}) \leqslant N$. Substituting (C.24), (C.25) into (C.23), using (C.17) and changing $\mu \to \mu^{\mathrm{tr}}$ in the summation then gives

$$
Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \sum_{\substack{\lambda,\mu \\ \ell(\lambda),\ell(\mu)\leqslant N}} s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) \left(\prod_{a=1}^{N} \int \mathrm{d}\mu(x_i, y_i) \, \mathrm{e}^{V(x_i^{-1}, \bar{\mathbf{t}}^{(1)}) + V(y_i^{-1}, \bar{\mathbf{t}}^{(2)})} \right) s_\lambda([x]) s_\mu([y]).
$$
(C.26)

At this point, the calculation becomes identical to the one in section 3. Using the Cauchy–Littlewood formula

$$
\mathrm{e}^{V(x^{-1}, \bar{\mathbf{t}}^{(1)})} = \sum_{\substack{\nu \\ \ell(\nu)\leqslant N}} s_\nu([x^{-1}]) s_\nu(\bar{\mathbf{t}}^{(1)}), \qquad \mathrm{e}^{V(y^{-1}, \bar{\mathbf{t}}^{(2)})} = \sum_{\substack{\eta \\ \ell(\eta)\leqslant N}} s_\eta([y^{-1}]) s_\eta(\bar{\mathbf{t}}^{(2)}),
$$
(C.27)

the identities (see section 3)

$$
s_\nu([x^{-1}]) = (x_1 \cdots x_N)^{-\nu_1} s_{\tilde{\nu}}([x]), \qquad s_\eta([y^{-1}]) = (y_1 \cdots y_N)^{-\eta_1} s_{\tilde{\eta}}([y]),
$$
(C.28)

where $\tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_N)$, $\tilde{\eta} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_N)$ and

$$
\tilde{\nu}_i := \nu_1 - \nu_{N-i+1}, \qquad \tilde{\eta}_i := \eta_1 - \eta_{N-i+1}
$$
(C.29)

and the product formulae

$$
s_\lambda([x]) s_{\tilde{\nu}}([x]) = \sum_\alpha c_{\lambda\tilde{\nu}}^\alpha s_\alpha(x), \qquad s_\mu([y]) s_{\tilde{\eta}}([y]) = \sum_\beta c_{\mu\tilde{\eta}}^\beta s_\beta([y]),
$$
(C.30)

we obtain

$$
Z_N(\mathbf{t}, n, m, \bar{\mathbf{t}}) = \sum_{\substack{\lambda,\mu,\nu,\eta \\ \ell(\lambda),\ell(\mu),\ell(\nu),\ell(\eta)\leqslant N}} I_{\lambda\mu\nu\eta}(N, n, m) s_\lambda(\mathbf{t}^{(1)}) s_\mu(\mathbf{t}^{(2)}) s_\nu(\bar{\mathbf{t}}^{(1)}) s_\eta(\bar{\mathbf{t}}^{(2)}),
$$
(C.31)

where

$$I_{\lambda\mu\nu\eta}(N, n, m) = \sum_{\alpha,\beta} c_{\lambda\tilde{\nu}}^{\alpha} c_{\mu\tilde{\eta}}^{\beta} J_{\alpha\beta}^{(n,m)} \tag{C.32}$$

and, in terms of the bimoments B_{ij},

$$
\begin{aligned}
J_{\alpha\beta}^{(n,m)} &:= N! \det(B_{l_i+n,l'_j+m})|_{1 \leqslant i,j \leqslant N}, \\
l_i &:= \alpha_i - i + N - \nu_1, \qquad l'_i := \beta_i - i + N - \eta_1.
\end{aligned}
\tag{C.33}
$$

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